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On the formation of weak plane shock waves by impulsive motion of a piston

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The piston problem for a viscous heat-conducting gas is studied under the assumption that the piston Mach number ϵ is small. The linearized Navier–Stokes equations are found to be valid up to times of the order of ϵ^{-2} mean free times after the piston is set in motion, while at large times the solution is governed by Burgers's equation. Boundary conditions for the large-time solution are supplied by the matching principle of the method of inner and outer expansions, which is also used to construct a composite solution valid both for small and for large times.

1. Introduction

In the piston problem of gasdynamics, we consider an indefinitely long tube to be filled with a viscous, heat-conducting gas and fitted at one end with a moveable piston. Initially the system is at rest. The piston is then impulsively set into motion at constant speed into the gas. We seek to describe the resultant gas motion, which, neglecting the effects of the tube walls (or, alternatively, considering the piston to be infinite in its lateral dimensions), we approximate as one-dimensional.

A physical description of the gas motion in the piston problem was given by Lagerstrom, Cole & Trilling (1949) (to whose report we shall subsequently refer as L.C.&T.) and by Lighthill (1956) on the basis of continuum theory. Immediately after the piston is set in motion, they concluded, the action is dominated by viscous diffusion of the initially sharp gradients in the flow. That is, at first the viscous terms in the Navier–Stokes equations are much more important than are the non-linear convective terms. Eventually, the gradients become small enough for the waveform-steepening effects of the non-linear terms to become comparable in strength to the viscous effects. The balance achieved between these opposing tendencies results in the formation of a shock wave which ultimately propagates steadily into the gas.

In this paper, we seek to fill in the above physical picture with mathematical details; at present only the asymptotic behaviours of the solution for small and large time are known. Of course, the picture itself is valid only after the piston has been moving for a time large compared with the mean time spent by a gas molecule between collisions. That is, until the molecules which have struck the moving piston collide with the other molecules in sufficient numbers, the continuum-flow assumptions which underlie the work of L.C.&T. and Lighthill cannot be justified. However, according to Lighthill, the time required for shock formation is a decreasing function of the shock strength; weak shocks do not form for many free times. If the shock strength (as measured by the piston Mach number, for example) is ϵ, the number of collisions experienced by the average molecule before a shock forms is of order $1/\epsilon^2$. Therefore, since the time required for transition from collisionless to continuum flow—which is certainly of the order of one mean free time—differs so widely from the time required for shock formation, by restricting our interest to weak shocks we can analyse the two phenomena separately. Here we choose to leave out the kinetic-theoretic effects.†

The exact Navier–Stokes equations and boundary conditions for the piston problem are given in §2. These are linearized and solved by Laplace-transform techniques in §3. From the physical picture sketched above, we expect this linearized solution to be valid for relatively small times. Examination of the solution shows, as noted above, that the linearization breaks down after about ϵ^{-2} mean free times. This suggests a set of stretched co-ordinates in which the final stages of shock formation can be studied. Re-expansion of the equations in the new co-ordinates verifies the conclusion reached by previous investigators that the solution at large times is governed by Burgers's equation. Boundary conditions on the large-time solution are now established by matching with the small-time solution, in the sense of the method of inner and outer expansions. The matching principle also yields a composite solution which is valid both for small and for large times. Our use of the matching principle is supported by analysis of a mathematical model of the piston problem as well as by the self-consistency of our results.

It should be noted that the linearized Navier–Stokes equations for the piston problem and the related 'shock-tube problem'‡ have previously been treated by Roy (1914), Cagniard (1941), Possio (1943), L.C.&T., and Sirovich (1964), while Wu (1956) and Ai (1960) studied the fundamental solutions which give the response to heat and momentum sources. However, because of difficulties encountered in the evaluation of certain integrals, none of these investigators gives more than the leading terms of large- and small-time expansions of the solution. In the present study, the complete expansions are found and evaluated.

It must also be noted that L.C.&T., Lighthill (1956), and Hayes (1960) have all shown the Burgers equation to govern the final stages of shock-wave formation. Our contribution here is the use of matched asymptotic expansions, which not only simplifies the derivation, but also connects the large- and small-time solutions, and so permits the determination of a uniformly valid composite solution.

† Moran (1965) attempted to calculate these effects on the basis of the linearized Krook equation. Laplace transforms were used to reduce the problem to the solution of certain pairs of simultaneous Weiner–Hopf equations, which unfortunately could not be inverted.

‡ That is, the boundary-free problem in which a one-dimensional flow starts from rest, with the density and pressure initially prescribed as step functions of the space co-ordinate.

2. Basic equations

We shall work in terms of dimensionless perturbation variables u, ρ, p, and T defined as follows:

$$\left.\begin{aligned} u^* &= \epsilon\sqrt{(RT_0^*)}\,u, \quad \rho^* = \rho_0^*(1+\epsilon\rho), \\ p^* &= p_0^*(1+\epsilon p), \quad T^* = T_0^*(1+\epsilon T). \end{aligned}\right\} \tag{2.1}$$

Here u^*, ρ^*, p^* and T^* are the dimensional fluid velocity, density, pressure, and temperature, respectively; R is the gas constant; and ϵ is a perturbation parameter, defined so that $u = 1$ on the piston (i.e. ϵ is proportional to the piston Mach number). The subscript zero denotes the initial (undisturbed) value of the variable. The longitudinal viscosity μ^*, defined by Hayes (1960) as $\frac{4}{3}$ the shear viscosity plus the bulk viscosity, is similarly non-dimensionalized:

$$\mu^* = \mu_0^*(1+\epsilon\mu). \tag{2.2}$$

The dimensionless independent variables x and t are defined so that

$$x^* \equiv \{\mu_0^*/\rho_0^*\sqrt{(RT_0^*)}\}\,x \tag{2.3}$$

is the distance into the fluid from the initial position of the piston, while

$$t^* \equiv (\mu_0^*/\rho_0^* RT_0^*)\,t \tag{2.4}$$

is the time after the piston is put into motion. It may be noted that this is tantamount to measuring distance and time in units of a molecule's mean free path and mean free time between collisions, respectively.

In this nomenclature, the exact Navier–Stokes versions of the conservation equations for mass, momentum, and energy in a one-dimensional unsteady flow may be written

$$\rho_t + u_x + \epsilon(\rho u)_x = 0, \tag{2.5}$$

$$u_t + p_x - u_{xx} + \epsilon[\rho u_t + u u_x - (\mu u_x)_x] + \epsilon^2 \rho u u_x = 0, \tag{2.6}$$

$$\begin{aligned} T_t + (\gamma-1)\,u_x - (\gamma/\sigma)\,T_{xx} + \epsilon[\rho T_t + uT_x + (\gamma-1)\,pu_x - (\gamma-1)\,u_x^2 - (\gamma/\sigma)\,(\mu T_x)_x] \\ + \epsilon^2[\rho u T_x + (\gamma-1)\,\mu u_x^2] = 0. \end{aligned} \tag{2.7}$$

Here γ is the specific-heat ratio and σ is the Prandtl number based on the longitudinal viscosity (which Hayes 1960 calls Pr''), both of which are assumed constant. We also have the perfect gas law

$$p = \rho + T + \epsilon\rho T. \tag{2.8}$$

The initial conditions for the piston problem are

$$u = \rho = p = T = 0 \quad \text{for} \quad x > 0, \quad t = 0, \tag{2.9}$$

while the boundary conditions at the piston, which we take to be impermeable and adiabatic, are

$$u = 1, \; T_x = 0 \quad \text{for} \quad t > 0, \; x = \epsilon t. \tag{2.10}$$

Finally, we impose damping conditions at infinity,

$$u, \rho, T \to 0 \quad \text{for} \quad t > 0 \quad \text{as} \quad x \to \infty. \tag{2.11}$$

3. The linearized solution

To begin with, we shall approximate the above equations by letting $\epsilon \to 0$ therein. The non-linear convective terms are then lost, but first approximations to the viscous terms are retained. According to the arguments quoted in § 1, the resulting solution may be expected to be good for small time. This we shall verify *a posteriori*. In the meantime, we shall give the variables associated with the linearized solution the superscript o (for 'outer'; cf. Van Dyke 1964).

Following L.C.&T., we find it convenient to start our analysis of the linearized versions of (2.5)–(2.11) by taking their Laplace transforms with respect to time. Indicating the operation by a bar over the transformed variable, e.g.

$$\bar{u}^o(x,s) = \int_0^\infty e^{-st} u^o(x,t)\,dt, \tag{3.1}$$

we obtain

$$s\bar{\rho}^o + \bar{u}^o_x = 0, \tag{3.2}$$

$$s\bar{u}^o + \bar{T}^o_x = (1+(1/s))\,\bar{u}^o_{xx}, \tag{3.3}$$

$$s\bar{T}^o + (\gamma-1)\,\bar{u}^o_x = (\gamma/\sigma)\,\bar{T}^o_{xx}, \tag{3.4}$$

$$\bar{p}^o = \bar{\rho}^o + \bar{T}^o, \tag{3.5}$$

$$\bar{u}^o = 1/s, \quad \bar{T}^o_x = 0 \quad \text{for} \quad x = 0, \tag{3.6}$$

$$\bar{u}^o, \bar{p}^o, \bar{T}^o \to 0 \quad \text{as} \quad x \to \infty. \tag{3.7}$$

Here we have already used (3.5) and (3.2) to eliminate $\bar{p}^o$ and $\bar{\rho}^o$ from (3.3).

Cross-differentiating (3.3) and (3.4), we find that $\bar{u}^o$ and $\bar{T}^o$ satisfy a fourth-order differential equation which may be put in the form

$$\left(\frac{\partial^2}{\partial x^2} - \lambda_1\right)\left(\frac{\partial^2}{\partial x^2} - \lambda_2\right)(\bar{u}^o \text{ or } \bar{T}^o) = 0. \tag{3.8}$$

Here

$$\lambda_i(s) = \frac{\sigma}{2}\frac{s}{1+s}\left\{1 + \left(\frac{1}{\gamma}+\frac{1}{\sigma}\right)s - (-)^i\left[1 + 2\left(\frac{1}{\gamma} - \frac{2}{\sigma\gamma} + \frac{1}{\sigma}\right)s + \left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)^2 s^2\right]^{\frac{1}{2}}\right\} \quad \text{for} \quad i = 1,2. \tag{3.9}$$

The general solution of (3.8) for $\bar{u}^o$ is of the form

$$\bar{u}^o = a_1(s)\,e^{-\surd(\lambda_1)x} + a_2(s)\,e^{-\surd(\lambda_2)x} + a_3(s)\,e^{+\surd(\lambda_1)x} + a_4(s)\,e^{+\surd(\lambda_2)x}. \tag{3.10}$$

From the damping condition (3.7), $a_3 = a_4 = 0$, while the piston boundary condition (3.6) requires $a_1 + a_2 = 1/s$. Thus

$$\bar{u}^o = a_1 e^{-\surd(\lambda_1)x} + \left(\frac{1}{s} - a_1\right) e^{-\surd(\lambda_2)x}. \tag{3.11}$$

Similarly, we find

$$\bar{T}^o = b_1 e^{-\surd(\lambda_1)x} - b_1 e^{-\surd(\lambda_2)x}. \tag{3.12}$$

To determine a_1 and b_1, we substitute (3.11) and (3.12) back into either (3.3) or (3.4) and equate the coefficients of both exponentials to zero. Thus we obtain

$$\bar{u}^o(x,s) = \frac{\sigma}{\gamma}\frac{1}{1+s}\frac{1}{\lambda_1-\lambda_2}\sum_{j=1}^{2}(-)^j e^{-\surd(\lambda_j)x}\left\{\gamma + s - \frac{\gamma}{\sigma}\frac{1+s}{s}\lambda_j\right\}, \tag{3.13}$$

$$\bar{T}^o(x,s) = \frac{\sigma}{\gamma}\frac{\gamma-1}{1+s}\frac{1}{\lambda_1-\lambda_2}\sum_{j=1}^{2}(-)^j e^{-\surd(\lambda_j)x}\frac{s}{\surd\lambda_j}. \tag{3.14}$$

Finally, from (3.2) and (3.13),

$$\bar{\rho}^o(x,s) = \frac{\sigma}{\gamma}\frac{1}{s(1+s)}\frac{1}{\lambda_1-\lambda_2}\sum_{j=1}^{2}(-)^j e^{-\sqrt{(\lambda_j)}x}\sqrt{\lambda_j}\left\{\gamma+s-\frac{\gamma}{\sigma}\frac{1+s}{s}\lambda_j\right\}, \quad (3.15)$$

which completes the solution in the Laplace world.

To return to the real world, we must evaluate contour integrals like

$$u^o(x,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\bar{u}^o(x,s)\,e^{st}ds, \quad (3.16)$$

where δ is real and sufficiently large that the singularities of the integrand lie to the left of $s = \delta$. As was the case with our predecessors, this inversion problem is severely complicated by the form of the equation (3.9) for the λ_i's which appear in (3.13)–(3.15). In the special case of unit longitudinal Prandtl number, however, the radical in (3.9) can be eliminated, and we find

$$\lambda_1 = s, \quad \lambda_2 = \frac{1}{\gamma}\frac{s^2}{1+s} \quad \text{for} \quad \sigma = 1. \quad (3.17)$$

Thus, we have developed complete small- and large-time expansions of the solution for $\sigma = 1$, but have worked out only the leading term of the large-time expansion in the case of arbitrary Prandtl number.

Details of the derivations are given elsewhere (Moran 1965). The general formulae for the asymptotic behaviour of the solution are found to be

$$\left.\begin{aligned} u^o(x,t) &\sim \tfrac{1}{2}\operatorname{erfc}[(x-\sqrt{\gamma}t)/\sqrt{(2\beta t)}]+O(t^{-\frac{1}{2}}),\\ \rho^o(x,t) &\sim \frac{1}{\sqrt{\gamma}}u^o(x,t)+O(t^{-\frac{1}{2}}),\\ T^o(x,t) &\sim \frac{\gamma-1}{\sqrt{\gamma}}u^o(x,t)+O(t^{-\frac{1}{2}}),\end{aligned}\right\} \quad (3.18)$$

where

$$\beta \equiv 1+(\gamma-1)/\sigma. \quad (3.19)$$

The small- and large-time expansions of the linearized solution for $\sigma = 1$ are rather lengthy, and so are relegated to the Appendix. From these expansions, time histories of the velocity, temperature and density profiles were constructed. The results for $\gamma = \frac{5}{3}$ are shown in figures 1–3.

The differences among the three profiles for small times contrast markedly with their similarity at large times. Also noteworthy is the definitely shock-like behaviour of the solution at large times. That is, as can also be seen from (3.18), the flow properties eventually exhibit a smooth transition between differing constant values, with the centre of the transition propagating at the speed of sound (which is $\sqrt{\gamma}$ in our system of measurement). Further, the values of the variables far from the transition zone satisfy the (linearized) Rankine–Hugoniot relations.†

† This should be expected, in spite of the erroneous shock structure predicted by linearized theory, since the Rankine–Hugoniot relations depend only on the existence of uniform flow outside the transition region, and are independent of the internal structure of the shock.

However, the width of the transition zone increases indefinitely with time like $\sqrt{t}$. This leads to an estimate of the time at which the linearized solution breaks down. As is well known, the thickness of a weak shock in steady flow is, on the

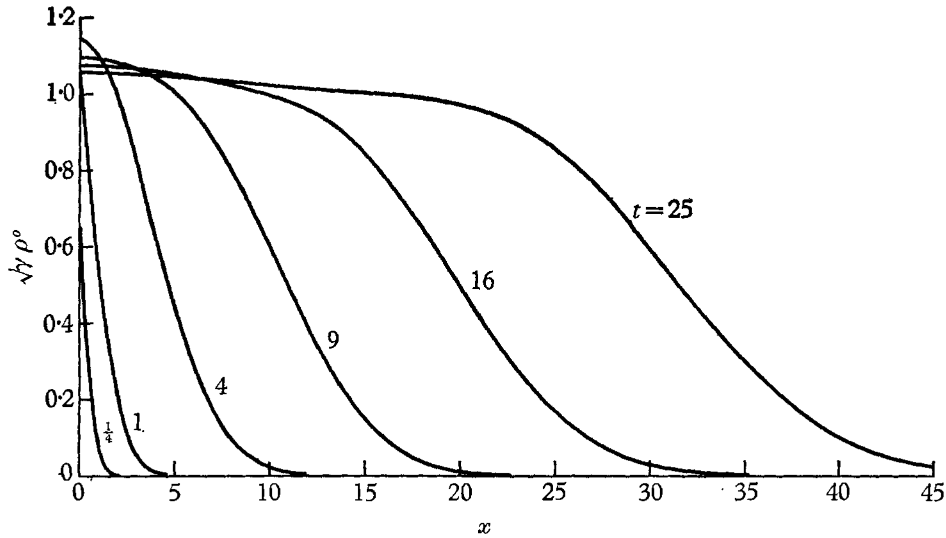

FIGURE 1. Density profiles in piston problem according to linearized Navier–Stokes equations.

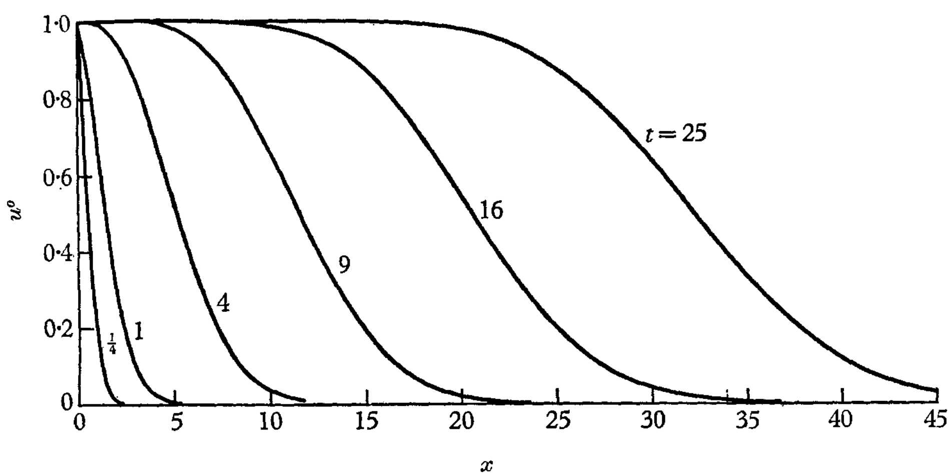

FIGURE 2. Velocity profiles in piston problem according to linearized Navier–Stokes equations.

present scale of measurement, of order $1/\epsilon$ (Taylor 1910). Since we expect the solution of the piston problem to yield a steady travelling shock as $t \to \infty$, we conclude that the linearized solution begins to break down when $t = O(1/\epsilon^2)$. For, when t is large compared to $1/\epsilon^2$, the linearized solution yields an over-thick shock.

A more formal argument leading to the same result may be constructed as follows. Underlying the linearization of equations (2.5)–(2.7) is the implicit assumption that ρ, u, T and their derivatives with respect to x and t are all of the same order of magnitude as $\epsilon \to 0$. Now *it is fairly easy to estimate a priori* the order of the variables themselves; with our system of non-dimensionalization, they are all of order unity. However, it is more difficult to estimate the relative

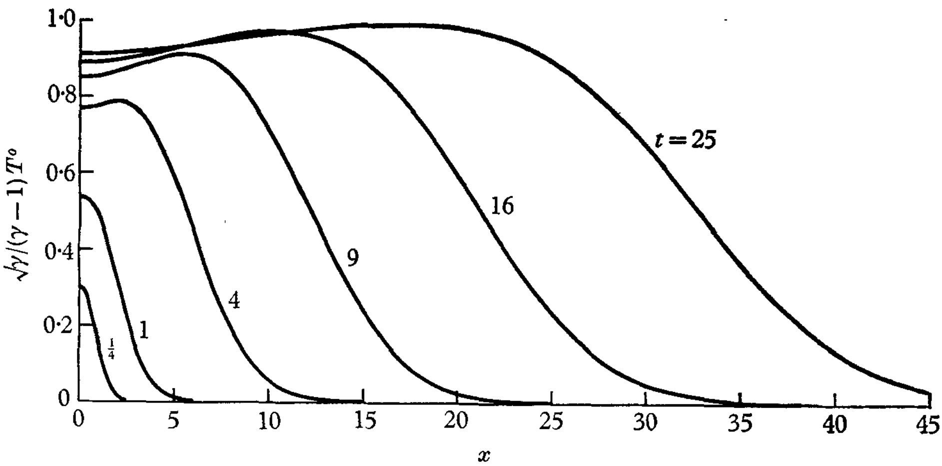

FIGURE 3. Temperature profiles in piston problem according to linearized Navier–Stokes equations.

size of the various derivatives. Though our choices for length and time scales are certainly not devoid of physical meaning, there is no reason to expect that, on these scales, the gradients of the variables are of any particular order in ϵ. *Thus this point must be investigated a posteriori.*

The investigation may be based on the asymptotic expressions (3.18) or, in the case $\sigma = 1$, on the complete expansions given in the Appendix. From these results, we conclude that the first derivatives of ρ^o, u^o, or T^o with respect to either x or t are of the same order for all t, but that the first two spatial derivatives of the unknowns change in relative order as t gets large; symbolically,

$$\frac{\partial u^o}{\partial t} = O\left(\frac{\partial u^o}{\partial x}\right), \tag{3.20}$$

for example, but

$$\frac{\partial^2 u^o}{\partial x^2} = O\left(\frac{1}{\sqrt{t}}\frac{\partial u^o}{\partial x}\right). \tag{3.21}$$

Thus, as $t \to \infty$, the order of magnitude of the viscous terms in (2.6) and (2.7) decreases relative to that of the non-linear inviscid terms. When

$$t = O(\epsilon^{-2}), \tag{3.22}$$

the two classes of terms are of the same order, and linearization can no longer be justified.

4. The solution at large times

The failure of linearization to yield a uniformly valid approximation to the solution suggests that the problem at large times must be attacked with singular-perturbation methods. First we introduce the nomenclature associated with such methods (Van Dyke 1964). The region of the (x, t)-plane in which $t \ll \epsilon^2$ is now dubbed the 'outer region', while the dependent and independent variables we have used heretofore become the 'outer variables'. The 'inner region' is the one in which the straightforward approach breaks down; viz. where t is comparable to or greater than ϵ^{-2}.

We begin our search for a solution approximately valid in the inner region by constructing a set of 'inner variables' whose variations in that region are of order unity as $\epsilon \to 0$. The dependent variables previously employed are already scaled properly, and so ρ, u and T will simply be given a superscript i to distinguish the inner solution for these quantities from the outer solution already obtained (to a first approximation). However, the independent variables x and t must be stretched. The appropriate transformation of the time co-ordinate is clearly

$$\tau = \epsilon^2 t. \tag{4.1}$$

For τ thus defined is of order unity when $t = O(\epsilon^{-2})$; i.e. when, according to (3.22), we are in the region where the linearized solution breaks down. In transforming the distance variable, we reason that, by the time the inner region is reached, the outer solution may be approximated by its asymptotic expansions (3.18). Thus only the interior of the developing shock wave is then of interest; outside everything is constant. Expecting this situation to hold true as t or $\tau \to \infty$, we therefore wish to centre the new distance co-ordinate with the shock. Further, it ought to be stretched by a factor of order ϵ so as to make the transformed shock thickness of order unity. Thus we let

$$\xi \equiv \epsilon(x - \surd(\gamma)\, t) \tag{4.2}$$

be the stretched length variable.

In these co-ordinates, the conservation equations (2.5)–(2.7) become

$$\epsilon[-\surd(\gamma)\,\rho^i_\xi + u^i_\xi] + \epsilon^2[\rho^i_\tau + (\rho^i u^i)_\xi] = 0, \tag{4.3}$$

$$\epsilon[-\surd(\gamma)\,u^i_\xi + \rho^i_\xi + T^i_\xi] + \epsilon^2[u^i_\tau + (\rho^i T^i)_\xi - u^i_{\xi\xi} - \surd(\gamma)\,\rho^i u^i_\xi + u^i u^i_\xi] = O(\epsilon^3), \tag{4.4}$$

$$\begin{aligned}&\epsilon[-\surd(\gamma)\,T^i_\xi + (\gamma-1)\,u^i_\xi]\\ &\quad + \epsilon^2[T^i_\tau - \gamma/\sigma\, T^i_{\xi\xi} - \surd(\gamma)\,\rho^i T^i_\xi + u^i T^i_\xi + (\gamma-1)\,\rho^i u^i_\xi + (\gamma-1)\,T^i u^i_\xi] = O(\epsilon^3),\end{aligned} \tag{4.5}$$

in which the magnitudes of the right sides are estimated on the assumption that our co-ordinate stretching has made all the derivatives of order unity.

We seek a first approximation to the solution of equations (4.3)–(4.5) as $\epsilon \to 0$. This task is complicated by the redundancy of the system to first order in ϵ; i.e., if we multiply (4.4) by $\surd\gamma$ and add to it (4.3) and (4.5), the terms of order ϵ cancel.

Sichel (1959, 1963) found similar redundancies in the equations which govern the structure of weak plane and curved shocks in steady flow. In each case, the explanation is similar. The severe stretching of the co-ordinates in the x-direction makes our transformed equations inviscid to first order, since the viscous terms consist of higher-order spatial derivatives. Were this the essence of the transformation, the first-order equations would be those of ordinary accoustics, and we could express their solution in terms of left- and right-running waves. Now if we stipulate that the solutions of the acoustic equations be a function only of $x-\sqrt{(\gamma)}\,t$, we find that they reduce to the first-order terms of (4.3)–(4.5). This suggests that, by centring the transformed co-ordinates with a right-running wave and stretching the time co-ordinate, we have effectively neglected left-running waves in the first approximation. This constitutes a constraint on the first-order solution, and so is responsible for the destruction of the linear independence of the first-order terms.

The only information contained in those terms is obtained by integrating them over ξ and applying the damping condition (2.11):

$$\sqrt{(\gamma)}\,\rho^i = \{\sqrt{\gamma}/(\gamma-1)\}\,T^i = u^i. \tag{4.6}$$

These relations show that the flow is isentropic in the first approximation, but are not sufficient to determine its evolution in time. Moreover, they admit, among others, solutions whose ξ-derivatives are large enough to invalidate the assumption that the terms in the square brackets of (4.3)–(4.5) are of order unity. Thus, following Sichel, we seek the additional relation required to complete the first approximation to the solution in the second-order terms of (4.3)–(4.5). We formally expand the solution in powers of ϵ — e.g. $u^i = u^i_0 + \epsilon u^i_1 + \ldots$ — and substitute into (4.3)–(4.5). Collecting terms of order ϵ, we obtain equations which are homogeneous and redundant in u^i_0, ρ^i_0, and T^i_0, and find on integration that those variables satisfy (4.6). The equations which result on collecting terms of order ϵ^2 in (4.3)–(4.5) are similarly redundant in u^i_1, ρ^i_1, and T^i_1, but are inhomogeneous. Thus they can have only trivial solutions unless a certain relation among the inhomogeneous terms holds, which leads us to conclude that

$$\begin{aligned}\epsilon^2[\rho^i_\tau + \sqrt{(\gamma)}\,u^i_\tau + T^i_\tau + u^i\rho^i_\xi + u^iT^i_\xi + (\gamma-1)\,T^iu^i_\xi \\ + \sqrt{(\gamma)}\,T^i\rho^i_\xi + \sqrt{(\gamma)}\,u^iu^i_\xi - \sqrt{(\gamma)}u^i_{\xi\xi} - \gamma/\sigma\,T^i_{\xi\xi}] = O(\epsilon^3)\end{aligned} \tag{4.7}$$

is the missing condition on the inner solution we have been seeking.†

Substitution of (4.6) into (4.7) shows that u^i satisfies Burgers's equation

$$u^i_\tau + \tfrac{1}{2}(\gamma+1)\,u^iu^i_\xi = \tfrac{1}{2}\beta u^i_{\xi\xi}, \tag{4.8}$$

in the first approximation, where β is defined in (3.19). This equation was regarded by Burgers (1948) as a mathematical model of the equations governing turbulence. He also noted that it could be used as a model equation for shock-wave formation, but it is seen here that the connexion between (4.8) and shock-wave theory is more definite than that.

† To be precise, what the indicated manipulations show is that u^i_0, ρ^i_0, and T^i_0 satisfy (4.7) without the $O(\epsilon^3)$ error term. Since we are only interested in a first approximation to the inner solution, we have dropped the subscript zero on those variables so as to avoid an over-cumbersome notation.

The role of Burgers's equation in the theory of shock-wave development has previously been shown by L.C.&T., Lighthill (1956) and Hayes (1960). Their derivations, however, are more complicated than the present one, as the use of stretched co-ordinates obviates the necessity for some rather intricate order-of-magnitude estimates.† Moreover, since they do not attempt to connect the large- and small-time solutions, they are not able to estimate the time at which (4.8) becomes valid, and they experience difficulties in ordering the x- and t-derivatives. L.C.&T., who treated only the infinite-Prandtl-number case, simply assumed that $u_t \ll u_x$. Noting the existence of a steady-state solution, they expected this approximation to improve as $t \to \infty$. Hayes made a similar approximation, while Lighthill made an order-of-magnitude analysis of sound-wave propagation.

Boundary conditions for equation (4.8) may be obtained by matching the inner and outer solutions in the sense of Kaplun & Lagerstrom (Van Dyke 1964). Symbolically, we assume that

$$(u^i)^o = (u^o)^i. \tag{4.9}$$

That is, if the inner solution is re-expressed in outer variables by means of (4.1) and (4.2) and re-expanded in ϵ for fixed x and t, the result ought to be the same as if the outer solution were put in inner variables and re-expanded for fixed ξ and τ. But, from (4.1), it is clear that, if we rewrite u^o in terms of ξ and τ and let $\epsilon \to 0$, we are effectively making an asymptotic expansion of u^o for large t. Therefore, from (3.18), (4.1), (4.2) and (4.9),

$$(u^i)^o = \tfrac{1}{2}\operatorname{erfc}\{\xi/\sqrt{(2\beta\tau)}\}. \tag{4.10}$$

Similarly, taking the outer limit of the inner solution is the same as letting $\tau \to 0$. Thus, from (4.10), we get an initial condition on the inner solution:

$$\begin{aligned} u^i(\xi,0) &= 0 \quad \text{for} \quad \xi > 0, \\ &= 1 \quad \text{for} \quad \xi < 0. \end{aligned} \tag{4.11}$$

Equation (4.11) shows that, in the final stages of shock formulation, the piston problem is indistinguishable from a corresponding shock-tube problem. This fact was deduced by Lighthill (1956) on physical grounds. He reasoned that 'the waveform gets away from the piston in a time negligible with the time scale of the process of shock-wave formation in which we are interested', so that one may as well study an initial-value problem as the corresponding boundary-value problem.

The general initial-value problem for Burgers's equation was solved by Hopf (1950) and Cole (1951); see also Lighthill (1956) and Hayes (1960). The key step is the introduction of a function $\psi(\xi,\tau)$, to which u^i is related by

$$u^i = -\frac{2\beta}{\gamma+1}\frac{\psi_\xi}{\psi}. \tag{4.12}$$

† While this paper was in preparation, J.-P. Guiraud informed the authors in private communication that he has shown Burgers's equation to govern the final stages of shock-wave formation in the shock-tube problem by employing techniques similar to those used here. Still another derivation, which is independent of the mechanism used to generate the shock, has been worked out by M. Lesser, who also uses co-ordinate stretching.

Substituting this into (4.8) and (4.11) and integrating over ξ, we find that ψ is a solution of the heat equation

$$\psi_\tau = \tfrac{1}{2}\beta\psi_{\xi\xi}, \tag{4.13}$$

subject to the initial conditions

$$\begin{aligned} \psi(\xi,0) &= \exp\left(-(\gamma+1)\,\xi/2\beta\right) \quad \text{for} \quad \xi < 0, \\ &= 1 \qquad\qquad\qquad\qquad\quad \text{for} \quad \xi > 0, \end{aligned} \tag{4.14}$$

where we have taken care to make the initial values continuous through $\xi = 0$. The solution can then be carried out by distributing heat sources along the ξ-axis (see, for example, Sommerfeld 1964), and, after substituting the result back into (4.12), we get

$$u^i(\xi,\tau) = \left[1 + \frac{\exp\{(\gamma+1)/2\beta\,(\xi - \frac{1}{4}(\gamma+1)\,\tau)\}\operatorname{erfc}\{-\xi/\surd(2\beta\tau)\}}{\operatorname{erfc}\{(\xi-\frac{1}{2}(\gamma+1)\,\tau)/\surd(2\beta\tau)\}}\right]^{-1}. \tag{4.15}$$

As noted above, the equation and initial conditions which govern Lighthill's (1956) solution of the piston problem are the same as (4.8) and (4.11). Thus, though our derivation differs greatly from his, (4.15) is also his result. As Lighthill notes, the solution is asymptotic to Taylor's (1910) result for the structure of a weak shock,

$$u^i \sim \left[1 + \exp\left\{\frac{\gamma+1}{2\beta}\epsilon(x-ct)\right\}\right]^{-1} \quad \text{as} \quad t \to \infty, \tag{4.16}$$

where

$$c \equiv \surd\gamma + \tfrac{1}{4}\epsilon(\gamma+1) \tag{4.17}$$

is the second approximation to the Rankine–Hugoniot result for the speed of a weak shock. Further, the time required for the solution to approach (4.17) to any specified degree of accuracy is clearly of order unity when expressed in the inner variables. Thus, the time required for the formation of a weak shock is of order ϵ^{-2} collision times, again as Lighthill (1956) noted.

If we now follow the prescription which accompanies (4.9), we find the leading term of the outer expansion of (4.15) to be

$$(u^i)^o = \tfrac{1}{2}\operatorname{erfc}\,(x - \surd(\gamma)\,t)/\surd(2\beta t). \tag{4.18}$$

Though (4.10) and (4.18) are identical, it should be noted that, in setting up the initial-value problem for the inner solution, we did not use (4.10) itself, but only its value at $\tau = 0$. In Van Dyke's (1964) nomenclature, we assumed only the 'limit matching principle', but have found *a posteriori* that our solution satisfies the more demanding 'asymptotic matching principle'. While the method of matched asymptotic expansions is in such wide use that perhaps we ought not to worry about the lack of a rigorous mathematical basis for the matching principle, this small check on our procedure is nevertheless comforting.

A further confirmation of our use of the matching principle was obtained by solving the system

$$v_t + (\epsilon v + c)\,v_x = \tfrac{1}{2}\delta v_{xx} \quad \text{for} \quad x > 0, \quad t > 0, \tag{4.19}$$

$$v(0,t) = 1 \quad \text{for} \quad t > 0, \tag{4.20}$$

$$v(x,0) = 0 \quad \text{for} \quad x > 0, \tag{4.21}$$

which may be regarded as a mathematical model of the piston problem†. That is, the term $\epsilon v v_x$ in (4.19) represents the non-linear convective terms in the Navier-Stokes equations which tend to steepen waves, while $\frac{1}{2}\delta v_{xx}$ represents the diffusive viscous terms, so that (4.19) contains terms which can be identified with the principal physical influences present in the real problem.

Since (4.19) is related to Burgers's equation (4.8) by a co-ordinate transformation, a substitution like (4.12) can be found which reduces (4.19) to the heat equation. The solution is facilitated by using Laplace transforms with respect to time, and the inversion of the transforms can be carried out in closed form without difficulty. By expanding $v(x, t)$ in powers of ϵ, we construct an outer expansion

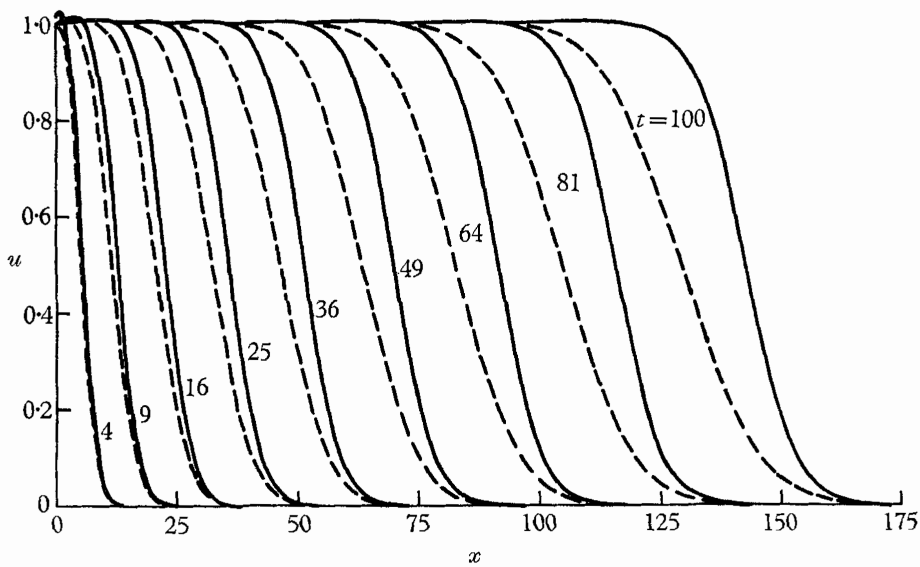

FIGURE 4. Velocity profiles in piston problem according to Navier–Stokes equations; – – – –, linearized solution; ——, composite solution ($\epsilon = 0{\cdot}2$).

whose leading term resembles (3.18). The inner expansion is obtained by putting the exact solution in terms of $\epsilon(x-ct)$ and $\epsilon^2 t$ before expanding in ϵ, and the leading term of the result looks like (4.15). Details of the solution are not important for present purposes. Here we need only note that the leading terms of the inner and outer expansions of the exact solution of the model problem satisfy the matching principle (4.9).

Because of the matching, the composite solution

$$u^c = u^i + u^o - (u^i)^o \tag{4.22}$$

is asymptotically valid as $\epsilon \to 0$ in both the inner and the outer regions, and, because of our indirect confirmation of the matching principle, we are confident that

† A model of the piston problem which may be obtained from ours by letting $c \to 0$ in (4.19) was solved by J. D. Cole (Lagerstrom 1964). However, his case is not as appropriate as ours for studying the behaviour of the solution for small ϵ. In particular, if $\epsilon = \delta = 0$, (4.19)–(4.21) yield a right-running 'acoustic wave' at $x = ct$, while in Cole's case the wave is coincident with the piston.

this solution is a uniformly valid first approximation for all time. The solution is illustrated in figure 4 for the case $\sigma = 1$, $\gamma = \frac{5}{3}$, and $\epsilon = 0{\cdot}2$.

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Appendix

Linearized solution for $\sigma = 1$

$$u^i(x,t) = \exp(-x^2/4t)\,\mathrm{Re}\left\{w\left(\sqrt{\left(\frac{\gamma t}{\gamma-1}\right)} - \frac{ix}{2\sqrt{t}}\right)\right\} + F_0, \tag{A 1}$$

$$\rho^i(x,t) = -\frac{\gamma-1}{\gamma}\exp(-x^2/4t)\,\mathrm{Im}\left\{w\left(\sqrt{\left(\frac{\gamma t}{\gamma-1}\right)} - \frac{ix}{2\sqrt{t}}\right)\right\} + \frac{1}{\sqrt{\gamma}}F_{-1}, \tag{A 2}$$

$$T^i(x,t) = \frac{\gamma-1}{\gamma}\exp(-x^2/4t)\,\mathrm{Im}\left\{w\left(\sqrt{\left(\frac{\gamma t}{\gamma-1}\right)}\frac{ix}{2\sqrt{t}}\right)\right\} + \frac{\gamma-1}{\sqrt{\gamma}}F_1. \tag{A 3}$$

Here
$$w(z) \equiv e^{-z^2}\,\mathrm{erfc}\,(iz) \tag{A 4}$$

while F_{-1}, F_0, and F_1 have the series expansions

$$F_l(x,t) = e^{-t}\sum_{m=1}^{\infty}(4t)^{m-\frac{1}{2}l}\sum_{n=0}^{m=1}\frac{1-(1-\gamma)^{n-m}}{n!}\left[\frac{x}{2\sqrt{(\gamma t)}}\right]^n i^{2m-n-l}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{(\gamma t)}}\right), \tag{A 5}$$

in which $i^q\,\mathrm{erfc}\,z$ is the qth iterated integral of the complementary error function (Abramowitz & Stegun 1964).

The asymptotic behaviour of the F_l as $t \to \infty$ may be obtained from

$$\begin{aligned} F_l(x,t) \sim \tfrac{1}{2}\,\mathrm{erfc}\,\phi + \frac{1}{2}\frac{\exp(-\phi)^2}{\sqrt{(2\pi t)}}\Bigg\{&\sum_{m=0}^{\infty} a_m^l (2t)^{-\frac{1}{2}m}(-2)^{-m}H_m(\phi) \\ &+ \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n}\sum_{q=0}^{p+1}\binom{n}{p}\frac{a_{m-n-1}^l}{q!\,(p+1-q)!} \\ &\times(-2)^{-m-p-q-1}H_{m+p+q+1}(\phi)\,2^q(2t)^{-\frac{1}{2}m}\phi^{p+1-q}\Bigg\}, \end{aligned} \tag{A 6}$$

in which
$$\phi \equiv (x - \sqrt{(\gamma)}\,t)/\sqrt{(2\gamma t)}. \tag{A 7}$$

$H_n(\phi)$ is the Hermite polynomial of nth degree (see, for example, Abramowitz & Stegun 1964), and the a_m^l are given by

$$\begin{aligned} a_{-1}^l &= -1 \quad \text{for} \quad l = -1, 0, 1, \\ a_n^{-1} &= 2^{-n-1} - 2\sqrt{(\gamma-1)}\,\mathrm{Im}\,[\{1 + i(\gamma-1)^{-\frac{1}{2}}\}^{-n-1}] \quad \text{for} \quad n \geqslant 0, \\ a_n^0 &= a_n^{-1} - a_{n-1}^{-1} \quad \text{for} \quad n \geqslant 0, \\ a_n^1 &= a_n^0 - a_{n-1}^0 \quad \text{for} \quad n \geqslant 0. \end{aligned} \tag{A 8}$$

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